

# Calculating the Fierz Transformation for Higher Orders

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We consider the higher-order Fierz transformation, which corresponds to expanding a product of  $\bar{\psi}\Gamma\psi$  terms into a sum of products of Dirac densities and currents. It is shown that the Fierz transformation can be obtained by solving a large system of linear equations with fractional complex coefficients, which is practical at least up to fourth power. © 2001 Academic Press

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## 1. THE PROBLEM

*Fierz transformation* [1] is a name given to the expression of a certain product of nondiagonal matrix elements of Dirac  $\Gamma$ -matrices as an expansion into products of diagonal matrix elements, such as

$$(\bar{a}\Gamma_i b)(\bar{b}\Gamma_j a) = \sum_{k,l=1}^{16} c_{kl}(\bar{a}\Gamma_k a)(\bar{b}\Gamma_l b). \quad (1)$$

Here  $\Gamma_i$  stands for one of the 16 Dirac matrices  $\{\mathbf{1}, \gamma_5, \gamma_\mu, \gamma_5\gamma_\mu, \sigma_{\mu\nu}\}$  constituting a linearly independent basis in the space of complex  $4 \times 4$  matrices. The matrix elements denote products in Dirac-index space only, i.e.,

$$(\bar{a}\Gamma_i b) = \bar{\psi}_a(\mathbf{r}, t) \Gamma_i \psi_b(\mathbf{r}, t). \quad (2)$$

The Fierz transformation is useful for expressing exchange matrix elements in terms of densities, currents, and other diagonal ones, which greatly eases their use in, for example, relativistic mean-field theories. For this reason it has been studied extensively, see, e.g., the papers by Y. Takahashi [2] or generalizations to  $SU(n)$  [3].

In the context of nonlinear self-coupling of meson fields, higher-order versions of the Fierz transformation have become of interest (for a review see [4]). If we express the order as the number of  $\Gamma$ -matrices involved, the above Eq. (1) is of second order, and in this paper we will be concerned with constructing expansions in third order,

$$(\bar{a}\Gamma_i b)(\bar{b}\Gamma_j c)(\bar{c}\Gamma_k a) = \sum_{l,m,n=1}^{16} c_{lmn}(\bar{a}\Gamma_l a)(\bar{b}\Gamma_m b)(\bar{c}\Gamma_n c), \quad (3)$$

and in fourth order,

$$(\bar{a}\Gamma_i b)(\bar{b}\Gamma_j c)(\bar{c}\Gamma_k d)(\bar{d}\Gamma_l a) = \sum_{m,n,p,q=1}^{16} c_{mnpq}(\bar{a}\Gamma_m a)(\bar{b}\Gamma_n b)(\bar{c}\Gamma_p c)(\bar{d}\Gamma_q d). \quad (4)$$

Here, a general computer-based solution method will be discussed, using either computer algebra systems such as *Mathematica* [5] or conventional programming, which allows exact manipulation of fractional complex numbers. Although this approach is quite general, it will be applied preferentially to the case of symmetrized matrix elements with only the matrix  $\mathbf{1}$  appearing in the expression to be expanded.

## 2. SYMMETRIZATION

In many applications, the wave function indices will all be summed over, so that it is sufficient to deal with an expression symmetrized over the indices. Thus, on the left-hand side of Eq. (1) in the second-order case we can write

$$\sum_{ab}(\bar{a}\Gamma_i b)(\bar{b}\Gamma_j a) = \frac{1}{2} \sum_{ab} [(\bar{a}\Gamma_i b)(\bar{b}\Gamma_j a) + (\bar{b}\Gamma_i a)(\bar{a}\Gamma_j b)], \quad (5)$$

and the right-hand side of Eq. (1) will be symmetrized exactly in the same way.

Using the notation  $\sum_{\{abc\dots\}}$  to refer to the sum over all permutations of the symbols  $a, b, c, \dots$ , we can reformulate the Fierz transformation problem for the symmetrized matrix element as

$$\sum_{\{ab\}}(\bar{a}\Gamma_i b)(\bar{b}\Gamma_j a) = \sum_{\{ab\}} \sum_{kl} c_{kl}(\bar{a}\Gamma_k a)(\bar{b}\Gamma_l b), \quad (6)$$

where the factor  $\frac{1}{2}$  has been dropped on both sides.

It is important to realize that because of the product structure, the right-hand side is symmetric under an exchange of the  $\Gamma$  matrices as well. It is thus useful to introduce a notation for symmetrized terms,

$$\Gamma_k \otimes \Gamma_l = \sum_{\{ab\}}(\bar{a}\Gamma_k a)(\bar{b}\Gamma_l b) = (\bar{a}\Gamma_k a)(\bar{b}\Gamma_l b) + (\bar{a}\Gamma_l a)(\bar{b}\Gamma_k b), \quad (7)$$

which can easily be generalized to higher order; for example, in third order the symmetrized problem becomes

$$\sum_{\{abc\}}(\bar{a}ib)(\bar{b}jc)(\bar{c}ka) = \sum_{l \leq m \leq n} c_{lmn} \Gamma_l \otimes \Gamma_m \otimes \Gamma_n, \quad (8)$$

with

$$\Gamma_l \otimes \Gamma_m \otimes \Gamma_n = \sum_{\{abc\}} (\bar{a}\Gamma_l a)(\bar{b}\Gamma_m b)(\bar{c}\Gamma_n c), \quad (9)$$

where symmetrization could equivalently be carried out in the indices  $l, m, n$  instead of  $a, b, c$ . Terms of fourth and higher orders are defined analogously.

### 3. ALGEBRAIC DETERMINATION OF THE EXPANSION

The second-order Fierz transformation as defined in Eq. (1) can be viewed as a system of equations obtained by comparing coefficients in the  $4^4 = 16^2$  dimensional space spanned by the spinors  $\psi_a, \psi_b, \bar{\psi}_a,$  and  $\bar{\psi}_b$ . The coefficients are given by the components of the  $\Gamma$ -matrices and thus can be expressed as complex integers. The unknowns  $c_{kl}$  are  $16 \times 16$  in number, so that we have exactly the right number of equations, and since the  $\Gamma$ -matrices form a basis for the  $4 \times 4$  complex matrices, the decomposition (1) is always possible.

The solution of this system of linear equations can be carried out using the standard Gauss elimination algorithm, provided that the coefficients are not treated in floating point arithmetic, but as exact complex fractions.

For third-order Fierz transformations, the dimension of the system of equations is  $16^3$  and it is  $16^4$  for fourth order; that is, the complexity in going from second order to fourth order is in the ratio 1:16:256. For the latter case, practical solution would require substantial computing resources, but fortunately in cases of practical interest the number of terms in the expansion can be reduced substantially by symmetry and invariance requirements. In the symmetrized case of the preceding section, for example, the dimension in fourth order is reduced by the number of permutations  $4!$ .

### 4. SELECTION OF THE TERMS IN THE EXPANSION

The expansion into products of the diagonal matrix elements of the  $\Gamma$ -matrices is always possible, but usually is not the most useful expression of the Fierz transformation. To see this, let us look at an important special case: that of identity matrices on the left-hand side. The decomposition problem thus is

$$(\bar{a}b)(\bar{b}a) = \sum_{jk} c_{jk} (\bar{a}\Gamma_j a)(\bar{b}\Gamma_k b), \quad (10)$$

or, in symmetrized form,

$$\sum_{\{ab\}} (\bar{a}b)(\bar{b}a) = \sum_{jk} c_{jk} \Gamma_j \otimes \Gamma_k. \quad (11)$$

Since the left-hand side is a Dirac scalar, this means that the right-hand side also can contain only scalar combinations of  $\Gamma$ -matrices. The only scalar combinations built out of products of two  $\Gamma$ -matrices are  $\mathbf{1} \otimes \mathbf{1}$ ,  $\gamma_5 \otimes \gamma_5$ ,  $\gamma_\mu \otimes \gamma^\mu$ ,  $\gamma_5 \gamma_\mu \otimes \gamma_5 \gamma^\mu$ , and  $\sigma_{\mu\nu} \otimes \sigma^{\mu\nu}$ , assuming the familiar index summation convention. The Fierz transformation problem in this case thus can be restated as (note that here because of the complete symmetry of all

terms, the symmetrization can be omitted)

$$\begin{aligned}
 (\bar{a}b)(\bar{b}a) = & c_1(\bar{a}a)(\bar{b}b) + c_2(\bar{a}\gamma_5 a)(\bar{b}\gamma_5 b) + c_3(\bar{a}\gamma_\mu a)(\bar{b}\gamma^\mu b) \\
 & + c_4(\bar{a}\gamma_5\gamma_\mu a)(\bar{b}\gamma_5\gamma^\mu b) + c_5(\bar{a}\sigma_{\mu\nu} a)(\bar{b}\sigma^{\mu\nu} b).
 \end{aligned}
 \tag{12}$$

Note that symmetrization works slightly differently in this case: the scalar products sometimes make certain index combinations appear repeatedly in the expansion, but it is still sufficient to include only one ordering of the  $\Gamma$  matrices in the symmetrized terms.

Eq. (12) corresponds to 256 equations for the five unknown coefficients. Clearly, most equations will be redundant; eliminating them from the statement of the problem, however, turns out to complicate the solution, but the high degree of redundancy provides a welcome check for completeness and consistency of the assumed decomposition.

### 5. SOLUTION FOR THE FOURTH-ORDER CASE

Equation (12) will be used to illustrate the method of solution. Actually, two different approaches were used depending on the complexity of the problem.

For second and third order a very simple but flexible approach was implemented in *Mathematica*. The spinors were expressed as four-component vectors containing symbols of the form

$$\begin{aligned}
 \psi_a \rightarrow & (a1, a2, a3, a4), & \bar{\psi}_a \rightarrow & (aa1, aa2, aa3, aa4), \\
 \psi_b \rightarrow & (b1, b2, b3, b4), & \bar{\psi}_b \rightarrow & (bb1, bb2, bb3, bb4).
 \end{aligned}
 \tag{13}$$

The scalar products with the  $\Gamma$ -matrices can then be evaluated straightforwardly, yielding a representation of Eq. (12) as a linear equation in the coefficients  $c_i$  with coefficients biquadratic in the spinor components. The symmetrization is done automatically. The coefficients are determined successively by choosing one term in which  $c_1$  occurs; if this is, e.g.,  $c_1 aa1 a2 bb2 b4$ , the coefficient of  $aa1 a2 bb2 b4$  is extracted from the equation, yielding a linear equation in the  $c_i$  alone, which is solved for  $c_1$ . This is inserted into the equation, reducing the number of unknown coefficients by one, and the process is repeated until all  $c_i$  have been found. If any terms are then left in the equation, the expansion was not complete.

While this is a very straightforward and not particularly efficient solution, the steps can be automated using built-in functions, and it is quite flexible, since new terms can be added by simply writing them down as symbolic expressions in the spinors and  $\Gamma$ -matrices. For the fourth order, however, this process turned out to be too inefficient, so that, as an alternative, a Fortran-90 code was developed that uses a data type for fractional complex numbers and straightforward Gauss elimination. Programming the individual terms, though, requires substantially more coding.

Specifically, the four independent components of each spinor are represented by a number 0, 1, 2, 3, corresponding to a two-bit integer. The four spinor indices are then combined into an 8-bit index, which indicates the index of the equation to which this combination contributes. For example, the expression  $(\bar{a}b)(\bar{b}a)$  has (as only one of the nonvanishing matrix elements) a value of one for the 3-component of spinors  $\bar{\psi}_a$  and  $\psi_b$  and the 2-component of spinors  $\bar{\psi}_b$  and  $\psi_a$ . If we arrange the index with  $\bar{\psi}_a, \psi_a, \bar{\psi}_b, \psi_b$  in order of

**TABLE I**  
**Terms and Expansion Coefficients Appearing**  
**in Second Order**

Term	Coefficient
$\mathbf{1} \otimes \mathbf{1}$	$\frac{1}{4}$
$\gamma_5 \otimes \gamma_5$	$\frac{1}{4}$
$\gamma_\mu \otimes \gamma^\mu$	$\frac{1}{4}$
$\gamma_5 \gamma_\mu \otimes \gamma_5 \gamma^\mu$	$-\frac{1}{4}$
$\sigma_{\mu\nu} \otimes \sigma^{\mu\nu}$	$\frac{1}{8}$

decreasing magnitude, the index for the equation will be

$$3 \times 4^3 + 2 \times 4^2 + 2 \times 4 + 3 = 11101011_2 = 235. \quad (14)$$

The program will loop through all spinor component combinations and add the generated coefficients at the corresponding position in the system of equations.

To help with the proper selection of terms in the expansion, the code checks whether any of the coefficients of the term being generated are nonzero, since symmetry may cause cancellation in a nonobvious way, and also whether a term is directly proportional to one previously generated.

The system of equations is then solved using Gauss elimination, where more general linear dependences will become apparent. If the elimination does not solve *all* of the equations, it is clear that the decomposition is not complete and this is then indicated.

## 6. RESULTS

As sample results, we give the decomposition of the symmetrized term in second, third, and fourth order for the case of identity matrices, as tables of terms with the corresponding expansion coefficients. For the higher-order terms, some remarks about new features are made.

### 6.1. Second Order

The result for second order is

$$\begin{aligned}
 (\bar{a}b)(\bar{b}a) = & \frac{1}{4}(\bar{a}a)(\bar{b}b) + \frac{1}{4}(\bar{a}\gamma_5 a)(\bar{b}\gamma_5 b) + \frac{1}{4}(\bar{a}\gamma_\mu a)(\bar{b}\gamma^\mu b) \\
 & - \frac{1}{4}(\bar{a}\gamma_5 \gamma_\mu a)(\bar{b}\gamma_5 \gamma^\mu b) + \frac{1}{8}(\bar{a}\sigma_{\mu\nu} a)(\bar{b}\sigma^{\mu\nu} b). \quad (15)
 \end{aligned}$$

This result is repeated in Table I in order to indicate the correct reading of the tables for higher order, where, however, a full symmetrization of both sides of the equation becomes necessary according to Eqs. (8) and (9).

### 6.2. Third Order

In third order, the symmetrization is no longer trivial as it was in the second-order case. It may be surprising that terms with  $\gamma_5 \sigma_{\mu\nu}$  must be included; these can be equivalently

**TABLE II**  
**Terms and Expansion Coefficients Appearing**  
**in Third Order**

Term	Coefficient
$\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$	$\frac{1}{16}$
$\mathbf{1} \otimes \gamma_5 \otimes \gamma_5$	$\frac{3}{16}$
$\mathbf{1} \otimes \gamma_\mu \otimes \gamma^\mu$	$\frac{3}{16}$
$\mathbf{1} \otimes \gamma_5 \gamma_\mu \otimes \gamma_5 \gamma^\mu$	$-\frac{3}{16}$
$\mathbf{1} \otimes \sigma_{\mu\nu} \otimes \sigma^{\mu\nu}$	$\frac{3}{32}$
$\gamma_\mu \otimes \gamma_5 \gamma_\nu \otimes \gamma_5 \sigma^{\mu\nu}$	$-\frac{3i}{8}$
$\gamma_5 \otimes \sigma_{\mu\nu} \otimes \gamma_5 \sigma^{\mu\nu}$	$\frac{3}{32}$

formulated using the identity

$$\gamma_5 \sigma_{\mu\nu} = \frac{i}{2} \epsilon_{\kappa\lambda\mu\nu} \sigma^{\kappa\lambda}, \tag{16}$$

but retaining the matrix  $\gamma_5$  makes the space-reversal properties of the terms more readily apparent. Note that either way our basis still consists of only 16 linearly independent matrices. The resulting transformation is given in Table II.

### 6.3. Fourth Order

In fourth order the number of possible terms becomes larger and it is difficult to see which are independent. Using the basic building blocks  $\mathbf{1}$ ,  $\gamma_5$ ,  $\gamma_\mu$ ,  $\gamma_5 \gamma_\mu$ ,  $\sigma_{\mu\nu}$ , and  $\gamma_5 \sigma_{\mu\nu}$  in all possible combinations fulfilling the condition of coupling to a Dirac scalar, which also implies an even number of  $\gamma_5$ -matrices, proves sufficient, but also leads to many dependent terms which are eliminated with the program's help. The final result is given in Table III.

## 7. OTHER APPLICATIONS

A similar method can be applied to nonsymmetric and nonscalar terms; in this case of course more terms will appear in the expansion. They can be constructed as before from the basic building blocks  $\mathbf{1}$ ,  $\gamma_5$ ,  $\gamma_\mu$ ,  $\gamma_5 \gamma_\mu$ ,  $\sigma_{\mu\nu}$ , and  $\gamma_5 \sigma_{\mu\nu}$  in a way that yields the desired Lorentz transformation properties.

As an example, the following decomposition was obtained:

$$\begin{aligned} (\bar{a} \sigma_{\mu\nu} b)(\bar{b} \gamma^\nu a) &= -\frac{1}{4}(\bar{a} \gamma^\nu a)(\bar{b} \sigma_{\mu\nu} b) - \frac{1}{4}(\bar{a} \sigma_{\mu\nu} a)(\bar{b} \gamma^\nu b) \\ &+ \frac{3i}{4}(\bar{a} \gamma_\mu a)(\bar{b} b) - \frac{3i}{4}(\bar{a} a)(\bar{b} \gamma_\mu b) \\ &+ \frac{3i}{4}(\bar{a} \gamma_5 \gamma_\mu a)(\bar{b} \gamma_5 b) + \frac{3i}{4}(\bar{a} \gamma_5 a)(\bar{b} \gamma_5 \gamma_\mu b) \\ &- \frac{1}{4}(\bar{a} \gamma_5 \gamma^\nu a)(\bar{b} \gamma_5 \sigma_{\mu\nu} b) + \frac{1}{4}(\bar{a} \gamma_5 \sigma_{\mu\nu} a)(\bar{b} \gamma_5 \gamma^\nu b). \end{aligned} \tag{17}$$

**TABLE III**  
**Terms and Expansion Coefficients Appearing in**  
**Fourth Order**

Term	Coefficient
$\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}$	$\frac{1}{64}$
$\mathbf{1} \otimes \mathbf{1} \otimes \gamma_5 \otimes \gamma_5$	$\frac{3}{32}$
$\mathbf{1} \otimes \mathbf{1} \otimes \gamma_\mu \otimes \gamma^\mu$	$\frac{3}{32}$
$\mathbf{1} \otimes \mathbf{1} \otimes \gamma_5 \gamma_\mu \otimes \gamma_5 \gamma^\mu$	$-\frac{3}{32}$
$\mathbf{1} \otimes \mathbf{1} \otimes \sigma_{\mu\nu} \otimes \sigma^{\mu\nu}$	$\frac{3}{64}$
$\mathbf{1} \otimes \gamma_5 \otimes \sigma_{\mu\nu} \otimes \gamma_5 \sigma^{\mu\nu}$	$\frac{3}{32}$
$\mathbf{1} \otimes \gamma_\mu \otimes \gamma_5 \gamma_\nu \otimes \gamma_5 \sigma^{\mu\nu}$	$-\frac{3i}{8}$
$\gamma_5 \otimes \gamma_5 \otimes \gamma_5 \otimes \gamma_5$	$\frac{1}{64}$
$\gamma_5 \otimes \gamma_5 \otimes \gamma_\mu \otimes \gamma^\mu$	$\frac{1}{32}$
$\gamma_5 \otimes \gamma_5 \otimes \gamma_5 \gamma_\mu \otimes \gamma_5 \gamma^\mu$	$-\frac{1}{32}$
$\gamma_5 \otimes \gamma_5 \otimes \sigma_{\mu\nu} \otimes \sigma^{\mu\nu}$	$\frac{3}{64}$
$\gamma_5 \otimes \gamma_\mu \otimes \gamma_5 \gamma_\nu \otimes \sigma^{\mu\nu}$	$-\frac{3i}{8}$
$\gamma_\mu \otimes \gamma^\mu \otimes \gamma_\nu \otimes \gamma^\nu$	$\frac{1}{64}$
$\gamma_\mu \otimes \gamma^\mu \otimes \gamma_5 \gamma_\nu \otimes \gamma_5 \gamma^\nu$	$-\frac{3}{32}$
$\gamma_\mu \otimes \gamma^\nu \otimes \gamma_5 \gamma_\mu \otimes \gamma_5 \gamma^\nu$	$\frac{1}{16}$
$\gamma_\lambda \otimes \gamma^\lambda \otimes \sigma_{\mu\nu} \otimes \sigma^{\mu\nu}$	$\frac{3}{64}$
$\gamma_\lambda \otimes \gamma^\mu \otimes \sigma_{\mu\nu} \otimes \sigma^{\lambda\nu}$	$-\frac{1}{16}$
$\gamma_5 \gamma_\mu \otimes \gamma_5 \gamma^\mu \otimes \gamma_5 \gamma_\nu \otimes \gamma_5 \gamma^\nu$	$\frac{1}{64}$
$\gamma_5 \gamma_\lambda \otimes \gamma_5 \gamma^\lambda \otimes \sigma_{\mu\nu} \otimes \sigma^{\mu\nu}$	$-\frac{3}{64}$
$\gamma_5 \gamma_\lambda \otimes \gamma_5 \gamma^\mu \otimes \sigma_{\mu\nu} \otimes \sigma^{\lambda\nu}$	$\frac{1}{16}$
$\sigma_{\mu\nu} \otimes \sigma^{\mu\nu} \otimes \sigma_{\kappa\lambda} \otimes \sigma^{\kappa\lambda}$	$\frac{3}{256}$
$\sigma_{\mu\nu} \otimes \sigma^{\lambda\nu} \otimes \sigma_{\lambda\kappa} \otimes \sigma^{\mu\kappa}$	$-\frac{1}{64}$

Those terms in which the exchange of indices  $a$  and  $b$  changes the sign will drop out in the symmetrized version of this result, which is

$$\begin{aligned}
 & \frac{1}{2} [(\bar{a}\sigma_{\mu\nu}b)(\bar{b}\gamma^\nu a) + (\bar{b}\sigma_{\mu\nu}a)(\bar{a}\gamma^\nu b)] \\
 &= -\frac{1}{4}(\bar{a}\gamma^\nu a)(\bar{b}\sigma_{\mu\nu}b) - \frac{1}{4}(\bar{a}\sigma_{\mu\nu}a)(\bar{b}\gamma^\nu b) \\
 & \quad + \frac{3i}{4}(\bar{a}\gamma_5\gamma_\mu a)(\bar{b}\gamma_5 b) + \frac{3i}{4}(\bar{a}\gamma_5 a)(\bar{b}\gamma_5\gamma_\mu b). \tag{18}
 \end{aligned}$$

## 8. SUMMARY

Once it is realized that constructing a Fierz transformation of any order essentially just means solving a linear system of equations with fractional complex coefficients, standard techniques of numerical analysis are sufficient to solve the problem. The calculation is now feasible up to fourth order, and the main work that remains is the expression of the

transformation through meaningful couplings of the  $\Gamma$  matrices, which still requires treating each transformation separately. Our work here demonstrates that tractable solutions may be possible for each of these cases. Commented *Mathematica* and Fortran-90 programs are available from the authors.

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### REFERENCES

1. M. Fierz, *Z. Physik* **104**, 553 (1937).
2. Y. Takahashi, *J. Math. Phys.* **24**, 1783 (1983).
3. H.-S. Zong, F. Wang, and J.-L. Ping, *Commun. Theor. Phys.* **22**, 479 (1994).
4. T. Hoch, D. Madland, P. Manakos, T. Mannel, B. A. Nikolaus, and D. Strottman, *Phys. Rep.* **242**, 253 (1994).
5. S. Wolfram, *The Mathematica Book*, 4th ed. (Cambridge University Press, Cambridge, MA, 1999).